Simultaneous Ascending Auctions with Complementarities and Known Budget Constraints*

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Abstract

We study simultaneous ascending auctions of identical objects when bidders are financially constrained and their valuations exhibit complementarities. We assume the budget constraints are known but the values for individual objects are private information, and characterize noncollusive equilibria.

The equilibria exhibit the exposure problem. The bidder with the highest budget is more reluctant to bid, because the bidder with the lowest budget may end up pursuing only one object, thus preventing the realization of complementarities. In some states of the world both objects are assigned to the 'poorer' bidder although that bidder has a lower valuation.

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1 Introduction

Since their introduction by the Federal Communication Commission in 1994 simultaneous ascending auctions have become a commonly used auction format for the sale of multiple objects. One reason for their popularity is that they allow bidders to adjust their bids between different objects as the auction progresses in light of the new information about the likelihood of obtaining different subsets of objects. This is particularly useful when significant subsets of the objects on sale tend to exhibit complementaries (see e.g. Szentes and Rosenthal [15]) and the bidders have a limited amount of money for the auction, so that increasing the bid on one object decreases the amount of money available for bidding on other objects. The practical importance of budget constraints in the FCC auctions of spectrum licenses has been pointed out, among others, by Salant [14].

The joint presence of complementarities and budget constraints makes the theoretical analysis of the equilibria of simultaneous ascending auctions particularly complex. The main issue appears to be the so-called 'exposure problem'. The expression is used to indicate that while bidding on an object above its stand-alone value may be justified by the hope of buying it as a component of a bundle whose total value is still above the total payment implied by the current prices, doing so exposes a bidder to the risk of ending up with having to buy that object alone, and thus earn a negative surplus. At least in some cases the exposure problem induces bidders to bid less aggressively than they would do by taking fully into account the value of complementarities that can be realized. Although the exposure problem has been identified in settings without budget constraints, in our setup it only appears when both complementarities and budget constraints are present.

In this paper we focus on the interplay between complementarities and budget constraints in simultaneous ascending auctions, and in order to make the analysis tractable we limit attention to the case of known budget constraints. This is a good approximation of reality in some instances, and it is a useful first step for the analysis of the case with privately known budgets.

To our knowledge, there is no paper that deals simultaneously with complementarities and budget constraints in ascending auctions. Instead, the literature has separately introduced complementarities or budget constraints in standard auction models.

The effect of complementarities in simultaneous auctions has been studied, among others, by Rosenthal and Wang [13], Krishna and Rosenthal [9], Szentes and Rosenthal [15], [16], Englmaier et al. [7], Fang and Parreiras [8] and Chakraborty [5]. In all these papers the auction formats that have been considered are not of the ascending type; rather, these papers have focused on variants of first-price or second-price auctions. Our work is more related to Albano et al. [1] and Zheng [17]. These papers consider ascending auctions in which some 'global' bidders have complementarities and want to pursue both objects while other 'local' bidders pursue a single object. The identity of global and local bidders is common knowledge, although the stand-alone values and the complementarities are private information. Neither Albano et al. [1] nor Zheng [17] consider the presence of budget constraints, as we do. Another important difference is that we consider ex ante identical bidders. By this we mean that the stand alone values and the complementarities are drawn from the same distributions.

The impact of budget constraints for various auction formats has been analyzed in a seminal paper by Che and Gale [6]. Benoît and Krishna [2] have studied the impact of budget constraints on sequential auctions with complete information. Finally, in Brusco and Lopomo [3] we have studied simultaneous ascending auction without budget constraints with heterogenous objects and either zero or large complementarities, and in Brusco and Lopomo [4] we have analyzed simultaneous ascending auctions with privately known budget constraints and homogeneous objects, but without complementarities.

We focus on 'noncollusive' equilibria, i.e. equilibria in which the bidders do not try to split the objects when budget constraints are not binding. Collusive equilibria may exist (depending on the distribution of values) but the presence of budget constraints adds little to their analysis, and we refer to the previous literature.

We always assume that stand–alone values are private knowledge, and analyze both the case in which complementarities are common and known and the case in which complementarities are private knowledge. In both cases the noncollusive equilibrium is essentially unique and has the same qualitative features. The most interesting effect is the presence of the exposure problem. It turns out that it is the bidder with the highest budget who is affected by the problem, since the 'poor' bidder may start demanding only one object when the budget becomes binding. Fearing exposure, the 'rich' bidder leaves the auction earlier than a 'poor' bidder of the same type. As a consequence, there are states of the world in which both objects are assigned to the 'poor' bidder despite the fact that her valuation for the objects is lower. We conclude that the presence of budget constraints cause inefficiencies both because of an obvious direct effect (i.e. the 'poor' bidder may be unable to get both objects when her values are higher because the budget is binding) and a more subtle strategic effect, the so-called exposure problem. Contrary to the direct effect, the strategic effects distorts efficiency against the 'rich' bidder.

The rest of the paper is organized as follows. In Section 2 we specify the rules of the auction and the assumptions on the bidders' preferences. In Section 3 we analyze the case in which the complementarity terms are common knowledge, strictly positive and identical for the two bidders, while the stand-alone values are private information, and describe the equilibrium. In Section 4 we study the case of privately known complementarities (as well as stand–alone values), characterizing an equilibrium which is qualitatively similar to the one found for the case of common complementarities. Section 6 contains concluding remarks, and an appendix collects the proofs.

2 The Model

There are two identical units of a good and two risk neutral bidders. Each bidder $i \in \{1,2\}$ is willing to pay v_i for a single unit, and $2(v_i + k_i)$, $k_i \geq 0$, for both units. We will refer to the variables v_i and k_i as bidder i's 'stand-alone value' and 'complementarity premium' respectively. (Note that k_i is the *per-unit* premium.) Bidder i's surplus when she obtains $n \in \{1,2\}$ objects and pays a total amount of m is

$$U_i(n, m | v_i, k_i) = \begin{cases} v_i - m, & \text{if} \quad n = 1, \\ \\ 2(v_i + k_i) - m, & \text{if} \quad n = 2. \end{cases}$$

The four variables (v_1, v_2, k_1, k_2) are distributed independently, with support $[0,1]^2 \times [\underline{k}, \overline{k}]^2$, where $\underline{k} \geq 0$. The stand-alone values v_1, v_2 are identically distributed, with c.d.f. F and differentiable density f, and the complementarity premia k_1, k_2 are identically distributed, with c.d.f. G and differentiable density g. The realization of the pair (v_i, k_i) is privately observed by bidder i before the beginning of the auction.

We assume that each bidder i has a fixed amount of money (budget) w_i for bidding, so that the total payment m cannot exceed the budget w_i. The budget levels w_1 and w_2 are common knowledge. To simplify the analysis without altering the substance of our results we assume than $1 < w_1, w_2$, so that any bidder can always bid up her stand-alone value (i.e. $v_i < w_i$, for all $v_i \in [0,1]$). However, bidder i is unable to bid up to her valuation for the two-unit bundle whenever $w_i < 2(v_i + k_i)$. It is convenient to introduce the notation $h_i \equiv \frac{w_i}{2}$, since this is the highest unit price that bidder i can pay when buying both objects. We assume that the two budgets are different, and without additional loss of generality we set $w_1 < w_2$.

The objects are sold using a 'simultaneous ascending clock auction' (SACA) working as follows. Each bidder is given two buttons. There is a single price which starts at zero and increases at constant speed, until at least one bidder lifts at least one button. The general idea is that lifting l buttons at price p means demanding l fewer units at any price *higher than* p, until additional buttons are lifted. Thus by lifting one button at price p, and the second button at $p' > p$, bidder i communicates that she is willing to pay up to $2p$ for two units, and up to p' for one unit.

Demand reduction is irreversible: once released a button cannot be pushed again. This is the simplest version of the *activity rules* that are often used in simultaneous ascending auctions[1].

The continuous time format of the auction requires care in the specification of some details, in order to make sure that the resulting game form is well defined. As usual, the technical problem is that sometimes a player wants to react to an action by her opponent as soon as possible, and this may create an 'open set problem'. In our case this happens, for example, when the price is increasing and bidder 2 wants to end the auction as soon as possible after bidder 1 lifts one button; that is, if bidder 1 lifts one button at time τ, then bidder 2 wants to lift one of her buttons at the lowest time t such that $t > \tau$. This problem has no solution because the constraint set

[1] In most of the FCC auctions used for the sale of spectrum licenses variations of the following basic rule were put in effect: 'a bidder that places eligible bids for n units at round t cannot place bids for more than n units at any subsequent round $t' > t$.' (Milgrom [12]).

is open. To get around this issue, we specify that when one[2] button is lifted for the first time, say by bidder 1, the price stops for an interval of time δ during which bidder 2 is given a chance to reduce her demand at the same price.[3] We also want to allow bidder 1 to react to bidder 2's reaction, and so on until both bidders choose to do nothing, before the price can resume its upward movement.

The formal specification of the rules is as follows. Suppose that bidder i is the first to lift exactly one button at time t, when the price is p_t. Then the price stops raising, and bidder $j \neq i$ is asked whether she wants to react by lifting one or two buttons. If j lifts any button the auction ends; otherwise bidder i is asked whether she wants to lift her second button. If she does, the auction ends; and if she does not, the price resumes its upward movement starting from p_t, with bidder i pushing one button and bidder j pushing two buttons. Thus the price may start moving again only after both bidders have had a chance to react to the status quo and have chosen to do nothing.

The allocation of the objects and the price are determined as follows. Let t denote the first time at which bidder i reduces demand, i.e. releases one or two buttons. If bidder i lifts two buttons, then her opponent $j \neq i$ buys both objects at unit price p_t, unless j also releases two buttons at t, in which case the tie is broken by assigning the two-item bundle to each bidder with probability $\frac{1}{2}$. If both bidders release exactly one button at time t, then each bidder buys one object at price p_t. Finally, if bidder i releases one button and j does nothing at t, the price stops for an interval of time δ, during which j is given the opportunity to react: if j releases one button, the auction ends with each bidder buying one unit at p_t; if j releases two buttons, then i buys both objects at p_t each;[4] and if j does nothing, then i is given a chance to release her second button in which case j gets both objects at p_t each, otherwise the price resumes its upward movement, starting from $p_{t+\delta} = p_t$.

In this last case, the auction ends as soon as any bidder releases another button: if bidder j releases one or two buttons then each bidder buys one object at the current price; if instead bidder i releases her second button,

[2] If more than one button is released, the auction ends.

[3] Zheng [17] has similar auction rules.

[4] This is because, as mentioned at the beginning of the section, lifting one button at p_t indicates willingness to buy both units for p_t each, and at most one unit for any price above p_t.

bidder j wins both objects.

A few remarks on the auction rules are in order. First, note that the buttons are not object-specific. Therefore this auction format is equivalent to one in which any bidder can resume bidding on any object, even if she has not done so continuously since the beginning of the auction, as long as her bidding activity (the number of objects on which she is bidding) does not increase. This format, with the activity rule, is intermediate between the case where exit on each object is irrevocable and the case with unrestricted reentry.

Second, the tie-breaking rules are designed to maximize the probability of realizing the complementarities, i.e. of assigning both objects to the same bidder. This is why the two-unit bundle is allocated to one (randomly chosen) bidder when both bidders reduce their demand to zero simultaneously. Also, when one bidder reacts to the demand reduction of another bidder lifting both buttons, both objects are allocated to the bidder who reduced demand to one. By maximizing the probability of assigning both objects to a single bidder we minimize the inefficiencies caused by the presence of potentially binding budget constraints. If the tie-breaking rules were changed the equilibria described in the next two sections would remain qualitatively the same, but would entail additional distortions from the efficient outcome.

In our analysis we rule out (weakly) dominated strategies, hence we focus on equilibria where each bidder reduces demand to zero only after the price becomes at least as large as the stand–alone value. Moreover, since our main goal is to study the impact that the simultaneous presence of budget constraints and complementarities has on the level of efficiency that can be achieved in the SACA, we focus on 'noncollusive' equilibria, i.e. equilibria in which the bidders bid 'straightforwardly' as much as possible. Equilibria with a collusive flavor, in which the bidders manage to coordinate on buying one object each for a low price, can be constructed as follows.[5] Suppose that bidder 1 reduces demand to one when the price is 0, implicitly inviting the opponent to split the objects. If this 'offer' is declined (i.e. if bidder 2 does not releases any of her buttons), the only optimal continuation strategy for bidder 1 is to bid on a single unit up to her value v_1, in which case bidder 2's optimal continuation strategy is to release one object at an optimally chosen

[5] We have studied collusive behavior in simultaneous ascending auctions in Brusco and Lopomo [3].

'stopping time'. For some distributions of v_i and k_i, splitting the objects at $p = 0$ is an equilibrium, and there can be similar equilibria inducing the splitting of the objects at prices below h_1.

We ignore collusive equilibria here because their existence does not hinge on the presence of budget constraints. Our goal is to identify the distortions from the first best that arise in the SACA when both budget constraints and complementarities are present. Thus we only consider 'noncollusive' strategies, according to which bidder i bids on both objects as long as $p \leq \min\{h_1, v_i + k_i\}$, i.e. unless either the implied payment is more than her value for the bundle – i.e. $2p > 2(v_i + k_i)$; or the price is above h_1, in which case bidder 1 cannot bid on both objects, as her budget constraint becomes binding.

We begin by establishing an easy benchmark: without budget constraints, the SACA has a 'bundling' equilibrium in which both objects always go to the bidder with the highest total value, thus implementing the efficient allocation.[6]

Proposition 1 *If $2\left(1 + \overline{k}\right) < w_i$ for each $i \in \{1, 2\}$, there exists a perfect Bayesian equilibrium in which bidder i wins both objects whenever $v_i + k_i > v_{-i} + k_{-i}$.*

In the equilibrium of Proposition 1 each bidder bids straightforwardly, demanding both objects when $p \leq v_i + k_i$ and zero otherwise. This is always feasible under the assumption that the type with the highest value for the bundle $2\left(1 + \overline{k}\right)$ can do so. Thus on the equilibrium path the bidders only compete for the two-unit bundle. To guarantee that demanding one unit is never a profitable deviation, we select beliefs that assign high probability to low values for any bidder who lifts only one button. Thus if bidder i releases only one button (an out of equilibrium action) her opponent will not accept to split the objects and will instead keep bidding on both objects up to her value, because she expects to pay a low price. This makes the deviation unprofitable.

[6] Chakraborty [5] discusses conditions under which various simultaneous (but non-ascending) auctions may have 'bundling equilibria', i.e. equilibria in which the strategies are such that a bidder either gets the bundle or nothing, so that the exposure problem does not arise.

The relevant implication of Proposition 1 here is that the sole presence of complementarities is not sufficient in our setting to generate any distortion from efficiency, as the exposure problem does not arise without the possibility of binding budget constraints.[7] Thus the distortions from the efficient outcome that we are going to find are due to the simultaneous presence of complementarities and potentially binding budget constraints.

These distortions are caused by three effects. First, there is an obvious direct effect due to the fact that in some cases the bidder with the higher value for the two-unit bundle cannot afford to pay for both objects, hence the objects end up being split. The bidder who suffers more from this effect is the one with the lower budget.

A second, more interesting effect, is caused by the exposure problem that is created for the high budget bidder, i.e. bidder 2. As we will see, there are cases in which, in any noncollusive equilibrium, bidder 2 will lift both buttons well before the total payment implied by the current price arrives at her value for the bundle, because she is afraid of ending up having to buy a single object and thus earning negative surplus. The loss of social surplus in this case is not due to the fact that the objects are split, but rather by the fact that the bundle may be assigned to the bidder with the lower value. Interestingly enough, the bidder who is hurt by this second effect is the one with the *higher* budget, because it is the 'poor' bidder who is the first to reduce demand to one unit, thus creating the exposure problem for the 'rich' opponent.

Finally there is a 'monopsony effect' that arises after bidder 1 lifts one button at h_1. (We will show that this happens in any noncollusive equilibrium). In this case bidder 2 faces a classic monopsonistic trade-off between quantity (one versus two units) and price, and thus will generally lift one button before the total payment implied by the price reaches her value for the two-unit bundle. Therefore the objects may end up being split even if bidder 2 is not budget constrained and has a higher value for the bundle.

Proceeding to the formal analysis, we define an *assignment rule* as a set of four functions $q = \left(q_i^{(1)}, q_i^{(2)}\right)$, $i \in \{1,2\}$, where $q_i^{(j)} : [0,1]^2 \times \left[\underline{k}, \overline{k}\right]^2 \to [0,1]$ specifies the probability that j objects are assigned to bidder i when the type profile is $\theta := (v_1, k_1, v_2, k_2)$. An assignment rule is feasible if, for

[7] The exposure problem would reappear however if the objects were heterogeneous, (e.g. if a bidder were only interested in one of the objects), even without budget constraints.

each profile θ, the allocation $q(\theta)$ is consistent with the fact that there are two objects for sale.

Our first proposition pertains to the direct effect that budget constraints have on the outcome. In any noncollusive equilibrium, a bidder can win both objects only if her budget is at least as large as twice her opponent's stand-alone value. This is because the opponent reduces demand to zero only after the price becomes larger than the stand-alone value.

Proposition 2 *If q is the assignment rule of an equilibrium outcome, then for each $i = 1, 2$,*

$$q_i^{(2)}(\theta) = 0 \quad \text{if } h_i < v_{-i}. \tag{1}$$

The restriction in (1) generates a loss of social surplus because it immediately implies that the objects must be split (hence the complementarities cannot be realized) whenever $h_i < v_{-i}$ for each $i \in \{1, 2\}$. Note that this is true in *any* equilibrium (with undominated strategies). Thus Proposition 2 identifies an upper bound on the level of efficiency that can be obtained.

As we will see next, this upper bound cannot be obtained by any equilibrium of the SACA. This is because of both the exposure problem and the monopsony effect, which induce bidder 2 to reduce her demand, to zero and one respectively, before the total payment implied by the price arrives at her value. It is worth pointing out that, while the exposure problem only arises in the presence of complementarities, the monopsony effect is most severe when bidder 2's complementarity premium k_2 is zero, as the incentive to buy the second unit increases with k_2. Thus the overall impact that the presence of complementarities has on the level of efficiency with budget constrained bidders is ambiguous: while they create the exposure problem, they also mitigate the monopsony effect.

To see how these two additional effects manifest themselves in all noncollusive equilibria, first observe that straightforward bidding pushes the price to h_1 when both bidders have sufficiently high values, i.e. $v_i + k_i \geq h_1$ for each i. Once the price arrives at h_1, bidder 1 must reduce her demand, as her budget constraint becomes binding. The next lemma establishes, that her only equilibrium continuation strategy is to reduce her demand to one, and then, if bidder 1 keeps demanding both objects, lift her second button before the price resume its upward movement or bid on one unit up to v_1, depending on whether her stand-alone value is below or above h_1.

Lemma 1 *In any noncollusive equilibrium, all types of bidder 1 with $h_1 < v_1 + k_1$ reduce their demand to one at h_1, and then lift their second button, if $v_1 \in (h_1 - k_1, h_1)$, and bid on one unit up to v_1, if $v_1 > h_1$.*

In light of Lemma 1 it is easy to characterize the set of all equilibrium continuation strategies for bidder 2. Recall that in equilibrium, once the price has arrived at h_1, it is common knowledge that $v_i + k_i \geq h_1$ for each $i = 1, 2$. After bidder 1 stops the price by lifting one button, bidder 2 can reduce her demand to zero before the price starts moving again, thus earning zero surplus, or do nothing until the price moves again and arrives at any level $p \in [h_1, h_2]$, and then lift one button. The latter strategy yields both objects at unit price v_1 if $v_1 < p$, and one object at price p if $v_1 > p$, hence an expected surplus of

$$V(p|v_2, k_2) \equiv \int_{h_1}^{p} 2(v_2 + k_2 - v_1)\, dG(v_1) + [1 - G(p)](v_2 - p),$$

where $G(p) \equiv \frac{F(p) - F(h_1)}{1 - F(h_1)}$. It is easy to see that, as $v_2 + k_2$ approaches h_1, the integral in the expression above goes to zero and the second term becomes negative, hence $V(p|v_2, k_2) < 0$ for all $p \in [h_1, h_2]$. Therefore there exists a set of types of positive measure with $v_2 + k_2 > h_1$ for whom it is optimal to lift both buttons before the price starts raising again. These are the types whose behavior is affected by the exposure problem. In this case bidder 1 wins both objects, and the resulting allocation is inefficient whenever $v_1 + k_1 < v_2 + k_2$.

For all other types of bidder 2, any equilibrium continuation strategy is characterized by an optimal 'stopping time'

$$p^*(v_2, k_2) \quad \in \arg\max_{p \in [h_1, h_2]} V(p|v_2, k_2), \tag{2}$$

which is often strictly below bidder 2's willingness to pay $v_2 + k_2$. To see this, note that the first derivative

$$\left.\frac{\partial V(p|v_2, k_2)}{\partial p}\right|_{p = v_2 + k_2} = G'(p)\, k_2 - [1 - G(v_2 + k_2)]$$

is negative for k_2 sufficiently small.

We now proceed to complete the characterization of the noncollusive equilibria. We will first present the case in which complementarities are known and identical, since most of the intuition can be obtained for this special case, and then turn to the general case in which both stand–alone values and complementarity premia are privately known.

3 Known and Identical Complementarities

In this section we discuss the noncollusive equilibrium of the SACA when it is common knowledge that the bidders have the same complementarity premium, i.e. $\underline{k} = \overline{k} = k > 0$. To simplify the exposition we will assume $h_1 - k > 0$; the discussion can be easily adapted to accommodate the case $h_1 - k < 0$ without significant changes. Since complementarities are identical, efficiency requires that both objects be assigned to the bidder with the highest stand-alone value v_i.

We define a pair of *partition strategies* as follows.

Player 1 (low budget player):

- types with $v_1 \in [0,\ h_1 - k]$ bid on both objects up to $v_1 + k$, and then reduce their demand to zero;
- types with $v_1 \in (h_1 - k, h_1]$ bid on both objects up to h_1, and then reduce their demand to one; if bidder 2 does not react, they reduce demand to 0;
- types with $v_1 \in (h_1, 1]$ bid on both objects up to h_1, and then bid on one object up to v_1.

Player 2 (high budget player):

- types with $v_2 \in [0,\ h_1 - k]$ bid on both objects up to $v_2 + k$, and then reduce their demand to zero;
- types with $v_2 \in (h_1 - k, v_2^*]$, where the threshold v_2^* is determined by equation (4) below, bid on both objects up to h_1 and then, when bidder 1 reduces demand to 1, react by reducing demand to 0;
- types with $v_2 \in (v_2^*, 1]$ bid on both objects up to h_1, and then, when bidder 1 reduces demand to 1, do not react; if bidder 1 does not lift her second button, keep their demand at 2 until an optimally chosen

'stopping time' $p^*(v_2)$ determined by (5), and then reduce demand to 1, thus ending the auction.

The strategy of the low–budget bidder can be loosely described as follows. First, types with $v_1 + k \leq h_1$ bid straightforwardly, demanding two objects until the price reaches $v_1 + k$ and then dropping both objects. All other types bid on both objects until the price reaches h_1 and watch bidder 2's reaction. If 2 reduces her demand to 0, they win both objects,

Second, types with $v_1 + k > h_1$ and $v_1 \leq h_1$ try to buy both objects until the price reaches h_1. At that point they lift one button, hoping that the opponent will leave the auction. If the opponent leaves the auction then bidder 1 gets both objects. If instead the opponent does not react, thus keeping the demand at 2, then bidder 1 leaves the auction, since the stand alone value is inferior to the price. Finally, types with $v_1 + k > h_1$ and $v_1 > h_1$ behave much in the same way, except that they keep trying buying a single object until the price reaches v_1. Notice that these are exactly the types that bidder 2 fears the most, since they are the ones who can get bidder 2 exposed.

Consider now the strategy of the high budget player. Again, types with $v_2+k \leq h_1$ bid straightforwardly. The set $(h_1 - k, v_2^*]$ is the most interesting one, since these are the types who fear the exposure problem and may leave the auction against an opponent with lower value. These types demand two objects until the price reaches h_1. At that point, bidder 1 reduces demand to 1, and bidder two has to decide how to react. To better understand the problem of bidder two, consider what happens if the bidder does not react (thus keeping both buttons pushed). In this case, either bidder 1 reacts by leaving the auction, or bidder 2 remains in the auction. The first case occurs when $v_1 \leq h_1$, and in that case bidder 2 obtains a utility of $2(v_2 + k - h_1)$. The second case occurs when $v_1 > h_1$. Notice that in this case the auction continues with bidder 1 having reduced demand to one, so that bidder 2 is forced to buy at least one object. This implies that in this case the highest expected utility that bidder 2 can attain is given by

$$V(v_2) \equiv \max_{p \in [h_1, h_2]} \int_{h_1}^{p} 2(v_2 + k - v_1)\, dG(v_1) + [1 - G(p)](v_2 - p), \qquad (3)$$

where $G(v_1) = F(v_1 | v_1 \geq h_1)$. If we define

$$\phi \equiv \Pr(v_1 \leq h_1 | h_1 - k < v_1)$$

then we conclude that the expected utility of continuing to push two buttons after the opponent has reduced demand to one is

$$H(v_2) \equiv \phi\left[2(v_2+k-h_1)\right]+(1-\phi)V(v_2).$$

This is a continuous and strictly increasing function of v_2, and it is easy to see that the function is strictly negative at $v_2 = h_1 - k$ and strictly positive at $v_2 = h_1$. The threshold v_2^* is the unique solution to the equation

$$H(v_2) = 0, \tag{4}$$

and it is an interior point of the interval $[h_1 - k, h_1]$. In fact, notice that bidder 2 can always obtain a utility of zero by lifting both buttons. Thus, bidders with $v_2 < v_2^*$ are better off by leaving the auction, since it is too costly for them to be exposed to the risk of buying a single object. On the other end, types in the set $(v_2^*, 1]$ obtain a strictly positive expected utility if they continue to compete in the auction.[8] At that point they have to decide at what price p^* to let the auction end, i.e. they have to decide the highest price that they are willing to pay in order to try to get both objects. This 'optimal stopping time' is obtained solving the problem.

$$\max_{p\in[h_1,h_2]} \int_{h_1}^{p} 2(v_2+k-v_1)\,dG(v_1)+[1-G(p)](v_2-p). \tag{5}$$

Notice that the types of bidder 2 such that $v_2 \in (h_1 - k, v_2^*]$ always give up both objects when the price reaches h_1 and the opponent reduces demand to one. This means that they will lose both objects against all types of bidder 1 in the interval $[h_1 - k, 1]$. If follows that whenever $h_1 - k < v_1 < v_2 < v_2^*$ the two objects go to bidder 1.

The next proposition establishes the existence of the equilibrium. Its proof consists in showing that it is possible to find a set of consistent beliefs for which the partition strategies are sequentially rational.

Proposition 3 *When k is known and identical across bidders there exists a perfect Bayesian equilibrium in which the bidders adopt the partition strategy.*

[8] Notice that it is never optimal for bidder 2 to react by reducing demand to one. Bidder 2 has always the option of keeping both buttons pushed, observe the reaction of bidder 1, and then reducing demand to one immediately when the auction restarts.

The *outcome* of the equilibrium described in Proposition 3 is as follows.

- When $\min\{v_1, v_2\} \leq h_1 - k$, the outcome is the same as the one of the the bundling equilibrium described in Proposition 1. The auction ends with bidder i buying both objects, and paying $v_{-i} + k_{-i}$ for each, whenever $v_i + k_i > v_{-i} + k_{-i}$.

- When $(v_1, v_2) \in (h_1 - k, 1] \times (h_1 - k, v^*)$, bidder 1 reduces demand to 1 when the price reaches h_1, and bidder 2 reacts by reducing demand to zero. Thus, bidder 1 buys both objects for h_1 each.

- When $(v_1, v_2) \in (h_1 - k, h_1] \times (v^*, 1]$, bidder 1 reduces demand to 1 when the price reaches h_1, bidder 2 keeps both buttons pushed, and then bidder 1 releases her second button. Thus bidder 2 buys both objects for h_1 each.

- When $(v_1, v_2) \in (h_1, 1] \times (v^*, 1]$ bidder 1 reduces demand to 1 when the price reaches h_1, bidder 2 keeps both buttons pushed, and bidder 1 continues to push one button. Thus, the price starts raising again and bidder 1 bids up to v_1 on a single object, while bidder 2 releases one button at her optimal stopping time $p^*(v_2)$ defined in (5). Thus the auction ends with bidder 2 buying both objects for v_1, if $v_1 < p^*(v_2)$, and with each bidder buying one object for $p^*(v_2)$ otherwise.

The value v^* depends on the parameters of the model. It is interesting to observe what happens to v^* when h_2 changes. When h_2 increases, the value $V(v_2)$ (weakly) increases for each value v_2, since the constraint set expands. Therefore the expected value of continuing the auction increases, and the value v^* (weakly) decreases. Thus, the lowest value of v^* is obtained by setting $h_2 = 1$ in the definition of $V(v_2)$ and then solving equation (4). The highest value of v^* is instead obtained when h_2 converges to h_1 from above. In this case it becomes pointless for bidder 2 to try to get both objects, since the tight budget will force bidder 2 to reduce demand to one very quickly after the price has passed h_1. In that case $V(v_2)$ converges to $[1 - G(h_1)](v_2 - h_1) = v_2 - h_1$, since $G(h_1)$ goes to 0. Therefore, v^* converges to

$$\overline{v}_2 = \frac{h_1 - \phi k}{1 + \phi},$$

We finally observe that

$$\frac{dH}{dk} = \frac{d\phi}{dk}\left[2\left(v_2 + k - h_1\right) - V\left(v_2\right)\right] + \phi 2 + V\left(v_2\right) + \left(1-\phi\right) 2G\left(p^*\right) > 0$$

since

$$\frac{d\phi}{dk} = \frac{1 - F\left(h_1\right)}{\left(1 - F\left(h_1 - k\right)\right)^2} f\left(h_1 - k\right).$$

It follows that $\frac{dv^*}{dk} < 0$. Since the magnitude of v^* is a measure of the severity of the exposure problem, we conclude that the exposure problem becomes less severe as the value of the complementarities increases. This is intuitive. When k becomes larger the expected cost of getting a single object remains constant, but the expected benefit of getting the bundle goes up. Thus, more types will be willing to take the risk of pursuing the bundle.

4 Privately Known Complementarities

In this section we show that an equilibrium with the same qualitative properties of the one described in section 3 can be found for the more general case in which both v_i and k_i are privately known. Again, for ease of exposition we assume that $\overline{k} > h_1$, but the case $\overline{k} \leq h_1$ can be easily accommodated.

The pair of *partition strategies* is now defined as follows.

Player 1 (low budget player).

- Types with $v_1 + k_1 < h_1$ bid on both objects up to $v_1 + k_1$ and then reduce their demand to zero. We call this *set A*.
- Types with $h_1 - k_1 < v_1 < h_1$ bid on both objects up to h_1 and then reduce their demand to one; if bidder 2 does not react, they reduce demand to 0. We call this *set B*.
- Types with $v_1 \in (h_1, 1]$ bid on both objects up to h_1, then bid on one object up to v_1. We call this *set H*.

Player 2 (high budget player).

- Types with $v_2 + k_2 < h_1$ bid on both objects up to $v_2 + k_2$ and then reduce their demand to zero. We call this *set C*.

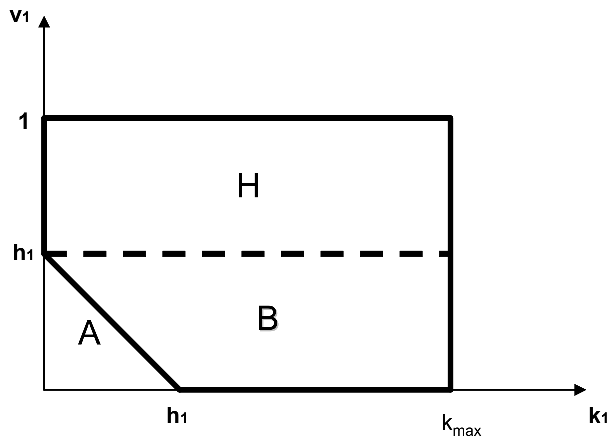

Figure 1: Partition of types for the low budget bidder.

- Types with $h_1 - k_2 < v_2 < v^{**}(k_2)$, where the function $v^{**} : [\underline{k}, \overline{k}] \to [0, h_1]$ is determined by equation (7) below, bid on both objects up to h_1 and then, when bidder 1 reduces demand to 1, react by reducing demand to 0. We call this *set D*.

- Types with $v^{**}(k_2) < v_2 < 1$ bid on both objects up to h_1, and then, when bidder 1 reduces demand to 1, do not react. If the opponent remains in the auction, they keep their demand at 2 until an optimal 'stopping time' $p^*(v_2, k_2)$ (determined by (6) below), and then reduce demand to one thus ending the auction. We call this *set G*.

The main difference with respect to the case in which complementarities are known and identical is that now the type sets are portions of the plane, rather than an interval. Figures 1 and 2 show how the partition strategy works for the low budget and high budget bidder, respectively.

The functions $v^{**}(k_2)$ is computed from an indifference condition, i.e. it gives the set of types (v_2, k_2) who are indifferent between not reacting when the opponent lifts one button at h_1 and lifting both objects. The formalism parallels that of the previous section. First notice that, once the low budget bidder has decided to remain in the auction, only the distribution of v_1 matters for bidder 2, and this is given by $G(v_1) = F(v_1 | v_1 \geq h_1)$.

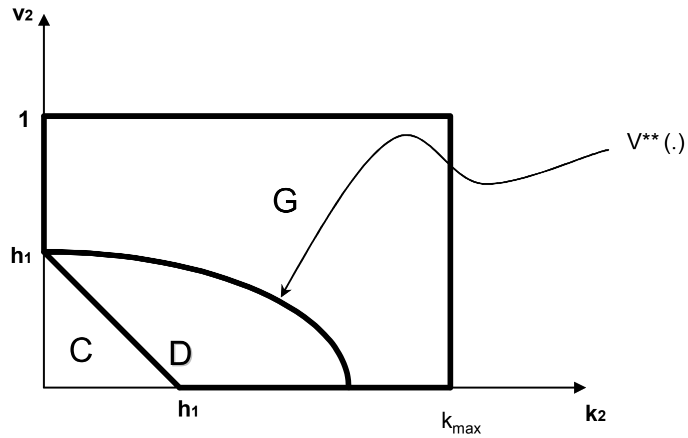

Figure 2: Partition of types for the high budget bidder.

Therefore, the optimal stopping time is found solving

$$\max_{p\in[h_1,h_2]} \int_{h_1}^{p} 2\left(v_2+k-v_1\right)dG\left(v_1\right)+\left[1-G\left(p\right)\right]\left(v_2-p\right), \tag{6}$$

and we will call $V\left(v_2,k_2\right)$ the value of the objective function at the optimal point. We also define

$$\xi \equiv \Pr\left(v_1 \leq h_1 | v_1+k_1 \geq h_1\right).$$

In graphical terms, the conditional probability ξ is the ratio between the probability mass contained in area B and the mass contained in the areas B and H of Figure 1. The expected utility for bidder 2 of continuing to push two buttons after the opponent has reduced demand to one is

$$H\left(v_2,k_2\right) \equiv \xi\left[2\left(v_2+k_2-h_1\right)\right]+\left(1-\xi\right)V\left(v_2,k_2\right).$$

For each fixed value k_2, the function $H\left(\cdot,k_2\right)$ is a continuous and strictly increasing function of v_2, strictly negative at $v_2=h_1-k_2$ and strictly positive at $v_2=h_1$. Thus, the value $v^{**}\left(k_2\right)$ is obtained as the unique solution to the equation

$$H\left(y,k_2\right)=0, \tag{7}$$

and it belongs to the interval $[h_1-k,h_1]$. The function $v^{**}\left(k_2\right)$ can be characterized as follows. First when $k_2=0$ we have $v^{**}\left(0\right)=h_1$. Second, the function is decreasing, since $\frac{\partial H}{\partial k_2} > 0$. Third, there is a value $\widehat{k}_2$ such that $H\left(0, \widehat{k}_2\right) = 0$. If $\widehat{k}_2 < \overline{k}$ then all types (v_2, k_2) with $k_2 \geq \widehat{k}_2$ will not react when bidder 1 lifts one button (this is the case pictured in Figure 2).

At this point, the task of completing the description of the perfect Bayesian equilibrium can be easily accomplished following the steps used in the proof of Proposition 3. We record this in the next proposition, which we state without proof.

Proposition 4 *When (v_i, k_i) is private information of bidder i there exists a perfect Bayesian equilibrium in which the bidders adopt the partition strategy.*

The *equilibrium outcome* can be described as follows.

- When $\min\{v_1 + k_1, v_2 + k_2\} \leq h_1$, i.e. either the type of bidder 1 is in A or the type of bidder 2 in is C, the outcome is the same as in the bundling equilibrium described in Proposition 1. The auction ends with bidder i buying both objects and paying a price $v_{-i} + k_{-i}$ whenever $v_i + k_i > v_{-i} + k_{-i}$.

- When $h_1 < v_1 + k_1 < 1$ and $h_1 - k_2 < v_2 < v^{**}(k_2)$, i.e. the type of bidder 1 is in B or H and the type of bidder 2 is in D, bidder 1 reduces demand to 1 when the price reaches h_1 and bidder 2 reacts reducing demand to zero. Thus bidder 1 buys both objects at a price h_1.

- When $h_1 - k_1 < v_1 < h_1$ and $v_2^*(k_2) < v_2 < 1$, i.e. the type of bidder 1 is in B and the type of bidder 2 is in G, bidder 1 reduces demand to 1 when the price reaches h_1. Bidder 2 does not react, and bidder 1 releases the second button. Thus bidder 2 buys both objects at h_1.

- When $h_1 < v_1 < 1$ and $v_2^*(k_2) < v_2 < 1$, i.e. the type of bidder 1 is in H and the type of bidder 2 is in G, bidder 1 reduces demand to 1 when the price reaches h_1, bidder 2 does not react, and bidder 1 bids up to v_1 on a single object. The auction continues until the price reaches v_1 (with bidder two getting both objects) or the optimal stopping time $p^*(v_2, k_2)$ (with the objects being split), whichever is lower.

The outcome of the equilibrium is inefficient basically in the same ways and for the same reasons described for the case of k known. One fact that should be pointed out however is that in the case of two-dimensional uncertainty there is no mechanism that can implement a ‘budget–constrained’ efficient allocation, i.e. an allocation that maximizes efficiency subject to the constraints given in Proposition 2, while in the case of one-dimensional uncertainty such mechanisms may exists. Since in this paper we do not deal with optimal mechanisms, we do not discuss the matter farther and refer the reader to Maskin [11] and Jehiel et al. [10].

5 An Example: The Uniform Case

To have a better grasp of the equilibrium, in this section we compute explicitly the equilibrium for the case in which v_i is uniformly distributed on $[0,1]$ and k_i is uniformly distributed on $\left[0,\overline{k}\right]$. For the low budget bidder we have

$$\Pr\left((v_1,k_1)\in A\right)=\frac{h_1^2}{2},\qquad \Pr\left((v_1,k_1)\in B\right)=\frac{(2-h_1)\,h_1}{2},$$
$$\Pr\left((v_1,k_1)\in H\right)=1-h_1.$$

Suppose that $\overline{k}$ is sufficiently large, so that we can conjecture that $v^{**}(k_2)$ intersects the horizontal axis a shown in Figure 2. We have

$$\xi=\frac{(2-h_1)\,h_1}{2-h_1^2}$$

The optimal stopping time is found solving

$$\max_{p\in[h_1,h_2]}\int_{h_1}^{p}2\left(v_2+k_2-v_1\right)dv_1+\left(v_2-p\right)(1-p)\,.$$

The first derivative of the objective function is v_2+2k_2-1, so that the optimal stopping time is h_1 when $v_2<1-2k_2$ or h_2 if $v_2>1-2k_2$ (and any point in the interval $[h_1,h_2]$ is optimal when $v_2=1-2k_2$). Therefore we have

$$V\left(v_2,k_2\right)=\begin{cases}\left(v_2-h_1\right)(1-h_1) & \text{if } v_2\leq 1-2k_2\\ \\ (1-h_1)\,v_2+(h_2-h_1)\left(v_2+2k_2\right)-h_2+h_1^2 & \text{if } v_2\geq 1-2k_2\end{cases}$$

and we observe that $V(v_2,k_2)$ is continuous and strictly increasing, since we have assumed $h_1<1$ and $h_2>h_1$. Notice that, as h_2 converges to h_1 from above, $V(v_2, k_2)$ converges to $(v_2 - h_1)(1 - h_1)$. The type $v^{**}(k_2)$ is found solving the equation

$$\xi 2 (v_2 + k_2 - h_1) + (1 - \xi) V(v_2, k_2) = 0$$

Consider first the interval $v_2 < 1 - 2k_2$. If $v^{**}(k_2)$ is in this interval then it must be the solution to

$$\xi 2 (v_2 + k_2 - h_1) + (1 - \xi)(v_2 - h_1)(1 - h_1) = 0$$

which is

$$v_2^{**}(k_2) = h_1 - \frac{2\xi}{(1 + \xi) - (1 - \xi) h_1} k_2. \tag{8}$$

If we define

$$T(h_1, \xi) = \frac{(1 - h_1)((1 + \xi) - (1 - \xi) h_1)}{2(1 - (1 - \xi) h_1)},$$

we have that the value will actually belong to the desired interval (i.e. $v_2^{**}(k_2)$ as defined in 8 is actually lower than $1 - 2k_2$) if

$$k_2 < T(h_1, \xi).$$

When k_2 exceeds this threshold the relevant equation becomes

$$\begin{gathered} \xi 2 (v_2 + k_2 - h_1) + \\ + (1 - \xi)\left(((1 - h_1) + (h_2 - h_1)) v_2 + 2(h_2 - h_1) k_2 - h_2 + h_1^2\right) = 0 \end{gathered}$$

and the solution is

$$v^{**}(k_2) = \frac{2\xi h_1 - (1 - \xi)\left(h_1^2 - h_2\right)}{2\xi + (1 - \xi)(h_2 + 1 - 2h_1)} - \frac{2(\xi + (1 - \xi)(h_2 - h_1))}{2\xi + (1 - \xi)(h_2 + 1 - 2h_1)} k_2$$

It can be readily checked that the solution belongs to the desired interval only if

$$k_2 > T(h_1, \xi).$$

We conclude that the function $v_2^{**}(k_2)$ has the following shape

$$v^{**}(k_2) = \begin{cases} h_1 - \frac{2\xi}{(1+\xi) - (1-\xi) h_1} k_2 & \text{if} \quad k_2 < T(h_1, \xi) \\ \\ \frac{2\xi h_1 - (1-\xi)\left(h_1^2 - h_2\right)}{2\xi + (1-\xi)(h_2 + 1 - 2h_1)} - \frac{2(\xi + (1-\xi)(h_2 - h_1))}{2\xi + (1-\xi)(h_2 + 1 - 2h_1)} k_2 & \text{if} \quad k_2 \geq T(h_1, \xi). \end{cases}$$

Since the absolute value of the slope is higher when k_2 is larger, in the uniform distribution case the types of the high budget bidder are partitioned as shown in Figure 3.

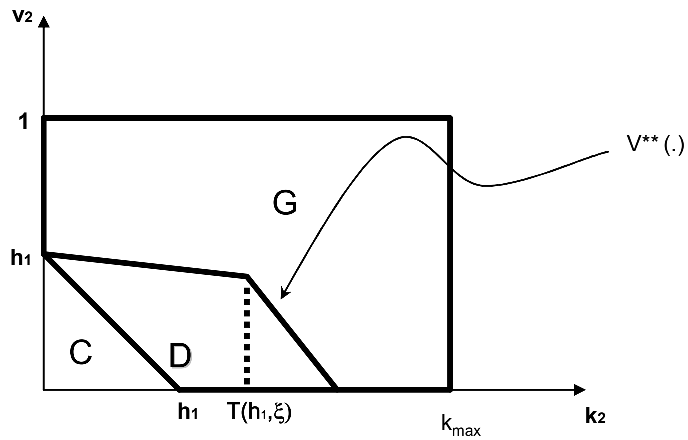

Figure 3: Partition of types for the high budget bidder in the uniform case.

6 Conclusions

In this paper we have discussed the structure of noncollusive equilibria in a simultaneous ascending clock auction in which the bidders are budget constrained and have increasing marginal payoffs from the objects. Our equilibria exhibit some intuitive properties, such as the existence of an exposure problem for the high-budget bidder. The simultaneous ascending clock auction has an efficient noncollusive equilibrium when there are no budget constraints, but it generates various inefficiencies when budget constraints are present. Not only objects are split too frequently, but it may also happen that, because of the exposure problem, the low–budget bidder may win the bundle even if her value for the bundle is lower than the value of the high budget bidder.

Appendix

Proof of Proposition 1. We specify strategies and beliefs, and show that they form a perfect Bayesian equilibrium. The equilibrium is symmetric, so the two bidders have the same strategies and beliefs.
Strategy. Type (v_i, k_i) of bidder i keeps both buttons pushed whenever the price is $p < v_i + k_i$, no matter how many buttons the opponent has previously released, and releases both buttons when the price reaches $p = v_i + k_i$. Also, the bidder releases all the remaining buttons whenever the price is $p > v_i + k_i$ (this can only happen out of equilibrium).
Beliefs. At any price p at which the opponent j has not released any button the belief on v_j and k_j is computed using the Bayes' rule, i.e. the belief on v_j is given by $F\left(v_j \middle| v_j + k_j \geq p\right)$ and the belief on k_j is given by $G\left(k_j \middle| v_j + k_j \geq p\right)$. If the opponent releases only one button at price p then the belief on v_j is any arbitrary distribution with support $[0, \min\{p, 1\}]$.

The beliefs are compatible with the strategy profile, since they are obtained using the Bayes' rule on the equilibrium path. We have to check optimality of the strategy on and off the equilibrium path.

If at any p bidder i keeps following the equilibrium strategy then the utility is $\max\{2\left[v_i + k_i - (v_j + k_j)\right], 0\}$. The only possible deviations are releasing two buttons, which gives 0, and releasing one object, which gives $\max\{\left[v_i - (v_j + k_j)\right], 0\}$. Thus, no deviation is profitable.

Out of the equilibrium path, the only case that matters is the one in which the opponent has released one button at some price $p' < p$, where p is the current price. Given the beliefs, the opponent will exit immediately the auction, so that by keeping both buttons pushed the utility is $2\left(v_i + k_i - p\right)$, since bidder i expects bidder j to leave the auction immediately. Releasing one or two buttons yields $(v_i - p)$, which is clearly less.

The outcome of this strategy profile is that both objects are sold to the bidder with the highest value for the two-unit bundle, say bidder i, for a price of $v_{-i} + k_{-i}$. Thus, the outcome is efficient. ■

Proof of Lemma 1. Once the price arrives at h_1, bidder 1 cannot continue to bid on both objects. Reducing demand to zero is dominated by reducing demand to 1, and lifting the second button if bidder 2 does not react. Thus, in any noncollusive equilibrium, all types of bidder 1 with

$v_1 + k_1 > h_1$ reduce their demand to one when the price arrives at h_1. If bidder 2 does not react, it is optimal for all types with $v_1 \in (h_1 - k_1, h_1)$ to lift the second button, as buying a single object for h_1 would generate a surplus of $v_1 - h_1 < 0$. For all types with $v_1 > h_1$ it is optimal to bid on one object up to v_1. ■

Proof of Proposition 3. We have to find beliefs which are compatible with the proposed strategy profile, describe behavior off the equilibrium path and show that no profitable deviation exists.

Beliefs are the same as in the proof of Proposition 1, i.e. they are given by the Bayes' rule on the equilibrium path and they assign low values to the stand-alone type of a bidder who drops only one object at any price lower than h_1.

The only relevant out-of-equilibrium path is the one in which some bidder i reduces demand to one at a price $p < h_1$. Given the beliefs, in that case it is optimal for the other player j to pursue both objects as long as $v_j + k_j > p$. The reason is that bidder i is expected to drop out of the auction immediately on the other object as well.

Finally, it is obvious (given the definition of optimal stopping time) that no profitable deviation exists for player 2 after bidder 1 has reduced demand to one at h_1. Also, no bidder can profit from reducing demand to one at a price $p < h_1$, since the opponent becomes more aggressive. In particular, notice that if bidder 2 reduces demand to one at $p < h_1$ then the opponent will try to get both objects until the price reaches $\min\{v_1 + k_1, h_1\}$. The outcome is therefore that bidder 2 gets only one object at a price $\min\{v_1 + k_1, h_1\}$, making the deviation unprofitable. ■

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